

Chebyshev polynomials and generalized complex numbers

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Abstract

The generalized complex numbers can be realized in terms of 2×2 or higher-order matrices and can be exploited to get different ways of looking at the trigonometric functions. Since Chebyshev polynomials are linked to the power of matrices and to trigonometric functions, we take the quite natural step to discuss them in the context of the theory of generalized complex numbers. We also briefly discuss the two-variable Chebyshev polynomials and their link with the third-order Hermite polynomials.

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We formulate the theory of Chebyshev polynomials [1] as a by-product of the generalized complex numbers [2]. To this aim we remind that any number h satisfying the identity

$$h^2 = a_2 + b_2 h \quad (1)$$

is a generalized complex number¹, whose higher-order powers are given by

$$h^n = a_n + h b_n \quad (n \in \mathbb{Z}^*) \quad (2)$$

with b_n, a_n obtained recursively ($a_0 = b_1 = 1, a_1 = b_0 = 0$). By multiplying both sides of this equation by h and equating the “real” and “imaginary” parts, i.e., the coefficients of the terms independent from and proportional to h , respectively, we obtain ($a_2 = a, b_2 = b$)

$$a_{n+1} = a b_n, \quad b_{n+1} = a_n + b b_n, \quad (3)$$

which can also be rewritten in the matrix form

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \hat{Q}(a, b) \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad \hat{Q}(a, b) = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}, \quad (4)$$

where the matrix \hat{Q} satisfies identity (2).

From Eq. (2) it also follows that

$$e^{h\phi} = \sum_{n=0}^{\infty} \frac{\phi^n}{n!} h^n = C(\phi) + h S(\phi), \quad (5)$$

with

$$C(\phi) = \sum_{n=0}^{\infty} \frac{\phi^n}{n!} a_n, \quad S(\phi) = \sum_{n=0}^{\infty} \frac{\phi^n}{n!} b_n, \quad (6)$$

that provides an Euler-like identity from which, taking into account Eq. (3), the following differential equations for the cos- and sin-like functions C and S , respectively, can be derived

$$\frac{d}{d\phi} C(\phi) = a S(\phi), \quad \frac{d}{d\phi} S(\phi) = C(\phi) + b S(\phi). \quad (7)$$

The complex unit h is characterized by the two conjugated forms (the solutions of Eq. (1))

$$h_{\pm} = \frac{b \pm \sqrt{b^2 + 4a}}{2} \quad (8)$$

¹ h reduces to the ordinary imaginary unit for $a_2 = -1, b_2 = 0$.

and, therefore,

$$h_{\pm}^n = a_n + h_{\pm} b_n, \quad (9)$$

from which we obtain

$$\begin{aligned} a_n &= \frac{h_+ h_-^n - h_- h_+^n}{h_+ - h_-} = \frac{a}{\sqrt{b^2 + 4a}} (h_+^{n-1} - h_-^{n-1}) \\ b_n &= \frac{h_+^n - h_-^n}{h_+ - h_-} = \frac{h_+^n - h_-^n}{\sqrt{b^2 + 4a}}. \end{aligned} \quad (10)$$

Analogously, we define the following Euler-like identity

$$e^{h_{\pm} \phi} = C(\phi) + h_{\pm} S(\phi) \quad (11)$$

from which we get

$$C(\phi) = h_+ e^{h_+ \phi} + h_- e^{h_- \phi}, \quad S(\phi) = \frac{e^{h_+ \phi} - e^{h_- \phi}}{h_+ - h_-}. \quad (12)$$

Let us now consider a recurrence of the type

$$L_{n+1} = a L_n + b L_{n-1}. \quad (13)$$

By using the Binet method [4], i.e., setting $L_n = h^n$, the solution of this difference equation can be written as

$$L_n = c_+ h_+^n + c_- h_-^n, \quad (14)$$

with c_{\pm} constants determined by the values of L_0 and L_1 . The Chebyshev polynomials satisfy the recurrence (13) with $a = 2x$ and $b = -1$ [1], and, therefore, to the corresponding difference equation we can associate the unit H defined by the equation

$$H^2 = 2xH - 1, \quad H_{\pm} = x \pm \sqrt{x^2 - 1} \quad (15)$$

and

$$H_{\pm}^n = A_n + H_{\pm} B_n, \quad (16)$$

where A_n and B_n are functions of the variable x satisfying recurrences that can be obtained by applying the same method used before (see Eqs. (3), (4)), which yield

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \hat{Q}(-1, 2x) \begin{pmatrix} A_n \\ B_n \end{pmatrix}. \quad (17)$$

Since $H_+ + H_- = 2x$, $H_+ H_- = 1$, from Eq. (16) it's easy to show that

$$\begin{aligned}\frac{H_+^n + H_-^n}{2} &= A_n + x B_n \\ H_+^n H_-^n &= A_n^2 + 2x A_n B_n + B_n^2 = 1.\end{aligned}\tag{18}$$

By replacing H_{\pm} to h_{\pm} in Eq. (10), by using again Eq. (15) one obtains

$$\begin{aligned}A_n &= \frac{H_+ H_-^n - H_- H_+^n}{H_+ - H_-} = -\frac{H_+^{n-1} - H_-^{n-1}}{2\sqrt{x^2 - 1}} \\ B_n &= \frac{H_+^n - H_-^n}{H_+ - H_-} = -A_{n+1}.\end{aligned}\tag{19}$$

Furthermore, since

$$\frac{dH_{\pm}}{dx} = \pm \frac{H_{\pm}}{\sqrt{x^2 - 1}}\tag{20}$$

we find that functions B_n verify the following differential equation

$$[(1 - x^2) \partial_x^2 - 3x \partial_x + (n - 1)^2] B_n(x) = 0\tag{21}$$

from which, by performing the replacement $n \rightarrow n+1$, we obtain the differential equation for the Chebyshev polynomials of second kind $U_n(x)$, and we can therefore make the following identification²

$$U_n(x) = \frac{H_+^{n+1} - H_-^{n+1}}{2i\sqrt{1 - x^2}}.\tag{22}$$

By specifying Eq. (2) to the case of matrix \hat{Q} for the Chebyshev polynomials, taking into account Eqs. (19), (13), one has

$$\hat{Q}^{n+1}(1, -2x) = A_{n+1} + \hat{Q}(1, -2x) B_{n+1} = \begin{pmatrix} -U_{n-1}(x) & -U_n(x) \\ U_n(x) & U_{n+1}(x) \end{pmatrix},\tag{23}$$

from which, since $\det \hat{Q}(1, -2x) = 1$, we get the well-known identity

$$U_n^2(x) - U_{n-1}(x) U_{n+1}(x) = 1.\tag{24}$$

It is well known that any 2×2 matrix can be written as a linear combination of the Pauli matrices and the unit matrix as follows [5]

$$\hat{M} = \alpha \hat{1}_2 + \sum_{k=1}^3 \beta_k \hat{\sigma}_k.\tag{25}$$

² The Chebyshev polynomials of first kind are given by $T_n(x) = (H_+^n + H_-^n)/2$.

As a consequence of the identity $\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{jk}$, we get

$$\hat{M}^2 = \gamma \hat{1}_2 + 2\alpha \hat{M} \quad (\gamma = -\alpha^2 + \sum_{k=1}^3 \beta_k^2), \quad (26)$$

and, therefore, according to previous point of view, any matrix \hat{M} can be viewed as the realization of a generalized complex unit. Furthermore, it can also be checked that [6, 7]

$$\hat{M}^n = U_{n-1}(\alpha) \hat{M} + U_{n-2}(\alpha) \hat{1}_2. \quad (27)$$

A step forward in the theory of Chebyshev polynomials has been done in [8] where their two-variable generalization $U_n^{(2)}(u, v)$ has been introduced. These polynomials satisfy the recurrence

$$U_{n+2}^{(2)} = u U_{n+1}^{(2)} - v U_n^{(2)} + U_{n-1}^{(2)} \quad (28)$$

and, therefore, are associated with the (third-order) imaginary unit defined by

$$Y^3 = u Y^2 - v Y + 1 \quad (29)$$

and to higher-order trigonometry discussed in Refs. [2] and [4].

We close the paper by noting that the generating function of the two-variable Chebyshev polynomials of second kind is given by [8]

$$\sum_{n=0}^{\infty} t^n U_{n+1}^{(2)}(u, v) = \frac{1}{1 - u t + v t^2 - t^3}. \quad (30)$$

The use of Laplace transform yields

$$\frac{1}{1 - u t + v t^2 - t^3} = \int_0^{\infty} ds e^{-s(1 - u t + v t^2 - t^3)} \quad (31)$$

and, therefore, on account of the generating function of the third-order Hermite polynomials

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(3)}(x, y, z) = e^{x t + y t^2 + z t^3}, \quad (32)$$

we get the following identity

$$U_{n+1}^{(2)}(u, v) = \frac{1}{n!} \int_0^{\infty} ds e^{-s} H_n^{(3)}(u s, -v s, s), \quad (33)$$

that can significantly simplifies the analysis of the properties of the two-variable Chebyshev polynomials, and of their multivariable generalization as well.

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